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ON THE MAPPING DEGREE SETS FOR 3-MANIFOLDS

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ABSTRACT. This note records the recent results on the following questions: Let M and N be a closed orientable 3-manifolds, $D(M, N)$ be the set of degrees of maps from M to N , denote $D(M, M)$ by $D(M)$.

(1) For which N , is the set $D(M, N)$ finite for any M ?

(2) If $D(M)$ is unbounded, what is $D(M)$?

(3) When is a self-map of degree ± 1 on M homotopic to a homeomorphisms?

Some of those results were presented at the RIMS Seminar at Akita Shirakami during September 13-17, 2010. For the proofs of those results, see [DeW2], [DeSW], [Wa1], [Du], [SWW], [SWWZ], [Sun].

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1. INTRODUCTION

Let M and N be two closed oriented 3-dimensional manifolds. Let $D(M, N)$ be the set of degrees of maps from M to N , that is

$$D(M, N) = \{d \in \mathbb{Z} \mid f: M \rightarrow N, \deg(f) = d\}.$$

We will simply use $D(N)$ to denote $D(N, N)$, the set of self-mapping degrees of N .

The calculation of $D(M, N)$ is a classical topic which often appeared in the literatures. According to [CT], Gromov thought it is a fundamental problem in topology to determine the set $D(M, N)$ for any dimension n .

Specially the calculation of $D(M)$, the integer set naturally associated to each closed orientable manifold M which presents an interesting connections between topology and number theory.

The result is simple and well-known for dimension $n = 1, 2$. For dimension $n > 3$, there are some interesting special results (See [DW] for recent ones and references therein), but it is difficult to get general results, since there are no classification results for manifolds of dimension $n > 3$.

The case of dimension 3 becomes the most attractive in this topic. Since Thurston's geometrization conjecture, which has been confirmed, implies that closed orientable 3-manifolds can be classified in reasonable sense.

A basic property of $D(M, N)$ is reflected in the following:

Question 1.1. (see [Wa2, Question 1.3] and [Re, Problem A]): *For which closed orientable 3-manifolds N , is the set $D(M, N)$ finite for any given closed oriented 3-manifold M ?*

It is clear if $D(N)$ is unbounded, then $D(M, N)$ is unbounded for some M . For each M , it is clear $\{0, 1\} \subset D(M)$, and if $D(M)$ is bounded then $D(M) \subset \{0, 1, -1\}$.

Question 1.2. *Let M be a closed orientable 3-manifold.*

- (1) *When is $D(M)$ bounded?*
- (2) *If $D(M)$ is unbounded, what is $D(M)$?*

Remark 1.3. The still unknown part for $D(M)$ is that if $D(M)$ is bounded, when does $-1 \in D(M)$?

The following related question is also natural and interesting.

Question 1.4. *For which closed orientable 3-manifolds M , whether there is a selfmap of degree ± 1 on M which is not homotopic to a homeomorphism on M ?*

Under Thurston's picture of 3-manifold, which is confirmed now, Question 1.2 (1) is answered 20 years ago; Question 1.1 and Question 1.2 (2) were answered very recently; the answer of Question 3 is known for Haken manifold and hyperbolic manifolds long times ago, and the answer is complete now for prime 3-manifolds. In Sections 2, 3 and 4, we will present those answers as well as how those answers are developed.

To end this section, we present the picture of 3-manifold which will be used to present the answers. All terminologies not defined are standard, see [He], [Sc] and [IR].

The picture of 3-manifolds: Each closed orientable 3-manifold N has a unique prime decomposition $N_1 \# \dots \# N_k$, the prime factors are unique up to the order and up to homeomorphisms. Each closed orientable prime 3-manifold N has a unique geometric decomposition such that each geometric piece supports one of the following eight geometries: H^3 , $\overline{PSL}(2, R)$, $H^2 \times E^1$, Sol, Nil, E^3 , S^3 and $S^2 \times E^1$ (where H^n , E^n and S^n are n -dimensional hyperbolic space, Euclidean space and sphere respectively), for details see [Th] and [Sc]. Moreover each geometric piece of N with non-trivial geometric decomposition supports either H^3 -geometry or $H^2 \times E^1$ -geometry, hence each 3-manifold supporting one of the remaining six geometry is closed. Furthermore each 3-manifold supporting geometries of either $H^2 \times E^1$, or E^3 , or $S^2 \times E^1$ is covered by a trivial circle bundle, and each 3-manifold supporting geometries of either Sol, or Nil, or E^3 is covered by a torus bundle. Call prime closed orientable 3-manifold N a *non-trivial graph manifold* if N has non-trivial geometric decomposition but contains no hyperbolic piece.

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2. ABOUT $D(M, N)$

This section is based on [DeW2] and [DeSW].

The answer of Question 1.1 is the following

Theorem 2.1. *Let N be a closed orientable 3-manifold. Then there is a closed orientable 3-manifold M such that $|D(M, N)| = \infty$ if and only if $|D(R)| = \infty$ for each prime factor R of N .*

In the following we will make a brief recall of the development of Theorem 2.1.

The development of Theorem 2.1: It is a common sense for many people that $|D(N)| = \infty$ for 3-manifold N which is either a product of a surface and the circle, or N is covered by the 3-sphere. The first significant result in this direction is due to Milnor and Thurston in the later 1970's. By using the minimum integer number of 3-simplices to build N [MT, Theorem 2], they proved

Theorem 2.2. *For each given hyperbolic 3-manifold N , $|D(M, N)| < \infty$ for any M .*

Gromov [G] introduced the simplicial volume $\|N\|$ for a manifold N , which is approximately the minimum real number of 3-simplices to build N . Gromov and Thurston proved that $\|N\|$ is proportional to the hyperbolic volume of N in the case of N is a hyperbolic 3-manifold, and then Soma proved $\|N\|$ is proportional to the sum of the hyperbolic volume of the hyperbolic pieces in the geometric decomposition of N (see [G], [Th], [So]). $\|*\|$ respects the mapping degrees, i.e. for any map $f: M \rightarrow N$ then $\|M\| \geq |\deg(f)| \cdot \|N\|$. Then it is deduced that

Theorem 2.3. *Suppose N is a closed orientable 3-manifold. If a prime factor of N has a hyperbolic piece in its geometric decomposition, then $|D(M, N)| < \infty$ for any M .*

Brooks and Goldman [BG1] [BG2] introduced the Seifert volume $SV(*)$ for closed orientable 3-manifolds which also respects the mapping degrees and is non-zero for each 3-manifold supporting the $\widetilde{PSL}(2, R)$ geometry. Then it is deduced that

Theorem 2.4. *Suppose N is a closed orientable 3-manifold. If a prime factor of N supports $\widetilde{PSL}(2, R)$ geometry. Then $|D(M, N)| < \infty$ for any M .*

Both Theorems 2.3 and 2.4 were already known in the early 1980's. The following result is known no later than early 1990's (see [Wa1] for example).

Theorem 2.5. *Suppose N is a closed orientable 3-manifold. Then $|D(N)| = \infty$ if and only if either N is covered by a torus bundle or a trivial circle bundle, or each prime factor of N is covered by S^3 or $S^2 \times E^1$.*

After Theorems 2.3, 2.4 and 2.5, the remaining unknown cases for Question 1.1 are: either a prime factor of N is a non-trivial graph manifold; or N is a non-prime 3-manifold, and $|D(R)| = \infty$ for each prime factor R of N , but some R is not covered by either S^3 or $S^2 \times E^1$.

In 2009 it is proved in [DeW2] that each closed orientable non-trivial graph manifold N has a finite covering \tilde{N} with positive Seifert volume (it is still unknown whether $SV(\tilde{N}) > 0$ implies $SV(N) > 0$ for a finite cover $\tilde{N} \rightarrow N$), and therefore it is deduced that

Theorem 2.6. *Let N be closed orientable non-trivial graph manifold. Then $|D(M, N)| < \infty$ for any closed orientable 3-manifold M .*

Remark 2.7. Two years before [DeW2], Theorem 2.6 is proved under the restriction that M are also graph manifolds [DeW1], by using a standard form of maps between graph manifolds [De1], and the estimation of the $PSL(2, R)$ -volume for a certain special class of graph manifolds.

In 2010 it is proved in [DeSW]

Theorem 2.8. *Let N be a given closed oriented 3-manifold N . If $|\mathcal{D}(R)| = \infty$ for each prime factor R of N , then there is a closed orientable 3-manifold M such that $|D(M, N)| = \infty$.*

Theorems 2.3 2.4, 2.6 and 2.8 (and Theorem 2.5) imply Theorem 2.1.

Remark 2.9. Theorem 2.8 follows from an explicit result [DeSW, Theorem 2.5], which provides the concrete M and the infinite set in $D(M, N)$ for the given N . The proof of Theorem 2.8 is essentially elementary, which does not appear until now mainly due to three reasons:

(1) $|\mathcal{D}(N)|$ may be finite even if $|\mathcal{D}(R)| = \infty$ for each prime factor R of N ; for example $|\mathcal{D}(T^3)| = \infty$ but $|\mathcal{D}(T^3 \# T^3)| < \infty$ for 3-dimensional torus T^3 [Wa1]. Such phenomena puzzled us to wonder if Theorem 2.8 was always true [Wa2, page 460].

(2) The target concerned in Theorem 2.8 became the only unknown case for Question 1.1 after the work [DeW2].

(3) The proof of Theorem 2.8 uses the result of $\mathcal{D}(N)$ which was just completely determined for each N recently ([Du], [SWW], [SWWZ]).

3. ABOUT $D(M)$

This section is based on [Wa1], [SWW], [SWWZ] and [Du].

3.1. Finer classes for calculate $D(N)$ when $D(N)$ is unbounded. To make this section to be complete, we allow it to have some light repeat with Section 2. The following result, which is a re-statement of Theorem 2.5, is known in early 1990's and answered Question 1.2 (1).

Theorem 3.1. *Suppose M is a geometrizable 3-manifold. Then M admits a self-map of degree larger than 1 if and only if M is either*

- (a) *covered by a torus bundle over the circle, or*
- (b) *covered by $F \times S^1$ for some compact surface F with $\chi(F) < 0$, or*
- (c) *each prime factor of M is covered by S^3 or $S^2 \times E^1$.*

Hence for any 3-manifold M not listed in (a)-(c) of Theorem 3.1, $D(M)$ is either $\{0, 1, -1\}$ or $\{0, 1\}$, which depends on whether M admits a self map of degree -1 . To determine $D(M)$ for geometrizable 3-manifolds listed in (a)-(c) of Theorem 1.0, let's have a close look of them.

For short, we often call a 3-manifold supporting Nil geometry a *Nil 3-manifold*, and so on. Among Thurston's eight geometries, six of them belong to the list (a)-(c) in Theorem 1.0. 3-manifolds in (a) are exactly those supporting either E^3 , or Sol or Nil geometries. E^3 3-manifolds, Sol 3-manifolds, and some Nil 3-manifolds are torus bundle or semi-bundles; Nil 3-manifolds which are not torus bundles or semi-bundles are Seifert fibered spaces having Euclidean orbifolds with three singular points. 3-manifolds in (b) are exactly those supporting $H^2 \times E^1$ geometry; 3-manifolds supporting S^3 or $S^2 \times E^1$ geometries form a proper subset of (3). Now we divide all 3-manifolds in the list (a)-(c) in Theorem 3.1 into the following five classes:

Class 1. M supporting either S^3 or $S^2 \times E^1$ geometries;

Class 2. each prime factor of M supporting either S^3 or $S^2 \times E^1$ geometries, but M is not in Class 1;

Class 3. torus bundles and torus semi-bundles;

Class 4. Nil 3-manifolds not in Class 3;

Class 5. M supporting $H^2 \times E^1$ geometry. We will present $D(M)$ for M in all those five classes. To do this, we need first to coordinate 3-manifolds in each class, then state the results of $D(M)$ in term of those coordinates. This is carried in the next subsection.

3.2. Main Results. Class 1. According to [Or] or [Sc], the fundamental group of a 3-manifold supporting S^3 -geometry is among the following eight types: \mathbb{Z}_p , D_{4n}^* , T_{24}^* , O_{48}^* , I_{120}^* , $T'_{8 \cdot 3^q}$, $D'_{n' \cdot 2^q}$ and $\mathbb{Z}_m \times \pi_1(N)$, where N is a 3-manifold supporting S^3 -geometry, $\pi_1(N)$ belongs to the previous seven ones, and $|\pi_1(N)|$ is coprime to m . The cyclic group \mathbb{Z}_p is realized by lens space $L(p, q)$, each group in the remaining types is realized by a unique 3-manifold supporting S^3 -geometry. Note also the sub-indices of those seven types groups are exactly their orders, and the order of the groups in the last type is $m|\pi_1(N)|$. There are only two closed orientable 3-manifolds supporting $S^2 \times \mathbb{E}^1$ geometry: $S^2 \times S^1$ and $RP^3 \# RP^3$.

Theorem 3.2. (1) $D(M)$ for M supporting S^3 -geometry are listed below:

$\pi_1(M)$	$D(M)$
\mathbb{Z}_p	$\{k^2 \mid k \in \mathbb{Z}\} + p\mathbb{Z}$
D_{4n}^*	$\{h^2 \mid h \in \mathbb{Z}; 2 \nmid h \text{ or } h = n \text{ or } h = 0\} + 4n\mathbb{Z}$
T_{24}^*	$\{0, 1, 16\} + 24\mathbb{Z}$
O_{48}^*	$\{0, 1, 25\} + 48\mathbb{Z}$
I_{120}^*	$\{0, 1, 49\} + 120\mathbb{Z}$
$T'_{8 \cdot 3^q}$	$\begin{cases} \{k^2 \cdot (3^{2q-2p} - 3^q) \mid 3 \nmid k, q \geq p > 0\} + 8 \cdot 3^q\mathbb{Z} & (2 \mid q) \\ \{k^2 \cdot (3^{2q-2p} - 3^{q+1}) \mid 3 \nmid k, q \geq p > 0\} + 8 \cdot 3^q\mathbb{Z} & (2 \nmid q) \end{cases}$
$D'_{n' \cdot 2^q}$	$\{k^2 \cdot [1 - (n')^{2^q-1}]^i \cdot [1 - 2^{(2p-q)(n'-1)}]^j \mid i, j, k, p \in \mathbb{Z}, q \geq p > 0\} + n'2^q\mathbb{Z}$
$\mathbb{Z}_m \times \pi_1(N)$	$\left\{ d \in \mathbb{Z} \mid \begin{array}{l} d = h + \pi_1(N) \mathbb{Z}, h \in D(N) \\ d = k^2 + m\mathbb{Z}, k \in \mathbb{Z} \end{array} \right\}$

(2) $D(S^2 \times S^1) = D(RP^3 \# RP^3) = \mathbb{Z}$.

Class 2. We assume that each 3-manifold P supporting S^3 -geometry has the canonical orientation induced from the canonical orientation on S^3 . When we change the orientation of P , the new oriented 3-manifold is denoted by \bar{P} . Moreover, lens space $L(p, q)$ is orientation reversed homeomorphic to $L(p, p - q)$, so we can write all the lens spaces connected summands as $L(p, q)$. Now we can decompose each 3-manifold in Class 2 as

$$M = (mS^2 \times S^1) \# (m_1 P_1 \# n_1 \bar{P}_1) \# \cdots \# (m_s P_s \# n_s \bar{P}_s) \\ \# (L(p_1, q_{1,1}) \# \cdots \# L(p_1, q_{1,r_1})) \# \cdots \# (L(p_t, q_{t,1}) \# \cdots \# L(p_t, q_{t,r_t})),$$

where all the P_i are 3-manifolds with finite fundamental group different from lens spaces, all the P_i are different from each other, and all the positive integer p_i are different from each other. Define

$$D_{iso}(M) = \{deg(f) \mid f : M \rightarrow M, f \text{ induces an isomorphism on } \pi_1(M)\}.$$

Theorem 3.3. (1) $D(M) = D_{iso}(m_1 P_1 \# n_1 \bar{P}_1) \cap \cdots \cap D_{iso}(m_s P_s \# n_s \bar{P}_s) \cap D_{iso}(L(p_1, q_{1,1}) \# \cdots \# L(p_1, q_{1,r_1})) \cap \cdots \cap D_{iso}(L(p_t, q_{t,1}) \# \cdots \# L(p_t, q_{t,r_t}));$
 (2) $D_{iso}(mP \# n\bar{P}) = \begin{cases} D_{iso}(P) & \text{if } m \neq n, \\ D_{iso}(P) \cup (-D_{iso}(P)) & \text{if } m = n; \end{cases}$
 (3) $D_{iso}(L(p, q_1) \# \cdots \# L(p, q_n)) = H^{-1}(C).$

The notions H and C in Theorem 3.3 (3) is defined as below:

Let $U_p = \{\text{all units in ring } \mathbb{Z}_p\}$, $U_p^2 = \{a^2 \mid a \in U_p\}$, which is a subgroup of U_p . We consider the quotient $U_p/U_p^2 = \{a_1, \dots, a_m\}$, every a_i corresponds with a coset A_i of U_p^2 . For the structure of U_p , see [IR] page 44. Define H to be the natural projection from $\{n \in \mathbb{Z} \mid \gcd(n, p) = 1\}$ to U_p/U_p^2 .

Define $\bar{A}_s = \{L(p, q_i) \mid q_i \in A_s\}$ (with repetition allowed). In U_p/U_p^2 , define $B_l = \{a_s \mid \#\bar{A}_s = l\}$ for $l = 1, 2, \dots$, there are only finitely many l such that $B_l \neq \emptyset$. Let $C_l = \{a \in U_p/U_p^2 \mid a_i a \in B_l, \forall a_i \in B_l\}$ if $B_l \neq \emptyset$ and $C_l = U_p/U_p^2$ otherwise. Define $C = \bigcap_{l=1}^{\infty} C_l$.

Class 3. To simplify notions, for a diffeomorphism ϕ on torus T , we also use ϕ to present its isotopy class and its induced 2 by 2 matrix on $\pi_1(T)$ for a given basis.

A *torus bundle* is $M_\phi = T \times I / (x, 1) \sim (\phi(x), 0)$ where ϕ is a diffeomorphism of the torus T and I is the interval $[0, 1]$. Then the coordinates of M_ϕ is given as below:

(1) M_ϕ admits E^3 geometry, ϕ conjugates to a matrix of finite order n , where $n \in \{1, 2, 3, 4, 6\}$;

(2) M_ϕ admits Nil geometry, ϕ conjugates to $\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, where $n \neq 0$;

(3) M_ϕ admits Sol geometry, ϕ conjugates to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $|a + d| > 2, ad - bc = 1$.

A *torus semi-bundle* $N_\phi = N \cup_\phi N$ is obtained by gluing two copies of N along their torus boundary ∂N via a diffeomorphism ϕ , where N is the twisted I -bundle over the Klein bottle. We have the double covering $p : S^1 \times S^1 \times I \rightarrow N = S^1 \times S^1 \times I / \tau$, where τ is an involution such that $\tau(x, y, z) = (x + \pi, -y, 1 - z)$.

Denote by l_0 and l_∞ on ∂N be the images of the second S^1 factor and first S^1 factor on $S^1 \times S^1 \times \{1\}$. A *canonical coordinate* is an orientation of l_0 and l_∞ , hence there are four choices of canonical coordinate on ∂N . Once canonical coordinates on each ∂N are chosen, ϕ is identified with an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $GL_2(\mathbb{Z})$ given by $\phi(l_0, l_\infty) = (l_0, l_\infty)$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

With suitable choice of canonical coordinates of ∂N , N_ϕ has coordinates as below:

(1) N_ϕ admits E^3 geometry, $\phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$;

(2) N_ϕ admits Nil geometry, $\phi = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix}$ or $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$, where $z \neq 0$;

(3) N_ϕ admits Sol geometry, $\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $abcd \neq 0, ad - bc = 1$.

Theorem 3.4. $D(M_\phi)$ is in the table below for torus bundle M_ϕ , where $\delta(3) = \delta(6) = 1, \delta(4) = 0$.

M_ϕ	ϕ	$D(M_\phi)$
E^3	finite order $k = 1, 2$	\mathbb{Z}
E^3	finite order $k = 3, 4, 6$	$\{(kt + 1)(p^2 - \delta(k)pq + q^2) \mid t, p, q \in \mathbb{Z}\}$
Nil	$\pm \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}, n \neq 0$	$\{l^2 \mid l \in \mathbb{Z}\}$
Sol	$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, a + d > 2$	$\{p^2 + \frac{(d-a)pr}{c} - \frac{br^2}{c} \mid p, r \in \mathbb{Z},$ either $\frac{br}{c}, \frac{(d-a)r}{c} \in \mathbb{Z}$ or $\frac{p(d-a)-br}{c} \in \mathbb{Z}\}$

(2) $D(N_\phi)$ is listed in the table below for torus semi-bundle N_ϕ , where $\delta(a, d) = \frac{ad}{gcd(a, d)^2}$.

N_ϕ	ϕ	$D(N_\phi)$
E^3	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	\mathbb{Z}
E^3	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\{2l + 1 \mid l \in \mathbb{Z}\}$
Nil	$\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}, z \neq 0$	$\{l^2 \mid l \in \mathbb{Z}\}$
Nil	$\begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix}$ or $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, z \neq 0$	$\{(2l + 1)^2 \mid l \in \mathbb{Z}\}$
Sol	$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, abcd \neq 0, ad - bc = 1$	$\{(2l + 1)^2 \mid l \in \mathbb{Z}\},$ if $\delta(a, d)$ is even or $\{(2l + 1)^2 \mid l \in \mathbb{Z}\} \cup \{(2l + 1)^2 \cdot \delta(a, d) \mid l \in \mathbb{Z}\},$ if $\delta(a, d)$ is odd

To coordinate 3-manifolds in Class 4 and Class 5, we first recall the well known coordinates of Seifert fibered spaces.

Suppose an oriented 3-manifold M' is a circle bundle with a given section F , where F is a compact surface with boundary components c_1, \dots, c_n with $n > 0$. On each boundary component of M' , orient c_i and the circle fiber h_i so that the product of their orientations match with the induced orientation of M' (call such pairs $\{(c_i, h_i)\}$ a section-fiber coordinate system). Now attach n solid tori S_i to the n boundary tori of M' such that the meridian of S_i is identified with slope $r_i = c_i^{\alpha_i} h_i^{\beta_i}$ where $\alpha_i > 0, (\alpha_i, \beta_i) = 1$. Denote the resulting manifold by $M(\pm g; \frac{\beta_1}{\alpha_1}, \dots, \frac{\beta_s}{\alpha_s})$ which has the Seifert fiber structure extended from the circle bundle structure of M' , where g is the genus of the section F of M , with the sign $+$ if F is orientable and $-$ if F is nonorientable, here 'genus' of nonorientable surfaces means the number of RP^2 connected summands. Call $e(M) = \sum_{i=1}^s \frac{\beta_i}{\alpha_i} \in \mathbb{Q}$ the Euler number of the Seifert fibration.

Class 4. If a Nil manifold M is not a torus bundle or torus semi-bundle, then M has one of the following Seifert fibreing structures: $M(0; \frac{\beta_1}{2}, \frac{\beta_2}{3}, \frac{\beta_3}{6})$, $M(0; \frac{\beta_1}{3}, \frac{\beta_2}{3}, \frac{\beta_3}{3})$, or $M(0; \frac{\beta_1}{2}, \frac{\beta_2}{4}, \frac{\beta_3}{4})$, where $e(M) \in \mathbb{Q} - \{0\}$.

Theorem 3.5. For 3-manifold M in Class 4, we have

- (1) $D(M(0; \frac{\beta_1}{2}, \frac{\beta_2}{3}, \frac{\beta_3}{6})) = \{l^2 | l = m^2 + mn + n^2, l \equiv 1 \pmod{6}, m, n \in \mathbb{Z}\};$
- (2) $D(M(0; \frac{\beta_1}{3}, \frac{\beta_2}{3}, \frac{\beta_3}{3})) = \{l^2 | l = m^2 + mn + n^2, l \equiv 1 \pmod{3}, m, n \in \mathbb{Z}\};$
- (3) $D(M(0; \frac{\beta_1}{2}, \frac{\beta_2}{4}, \frac{\beta_3}{4})) = \{l^2 | l = m^2 + n^2, l \equiv 1 \pmod{4}, m, n \in \mathbb{Z}\}.$

Class 5. All manifolds supporting $H^2 \times E^1$ geometry are Seifert fibered spaces M such that $e(M) = 0$ and the Euler characteristic of the orbifold $\chi(O_M) < 0$.

Suppose $M = (g; \frac{\beta_{1,1}}{\alpha_1}, \dots, \frac{\beta_{1,m_1}}{\alpha_1}, \dots, \frac{\beta_{n,1}}{\alpha_n}, \dots, \frac{\beta_{n,m_n}}{\alpha_n})$, where all the integers $\alpha_i > 1$ are different from each other, and $\sum_{i=1}^n \sum_{j=1}^{m_i} \frac{\beta_{i,j}}{\alpha_i} = 0$.

For each α_i and each $a \in U_{\alpha_i}$, define $\theta_a(\alpha_i) = \#\{\beta_{i,j} \mid p_i(\beta_{i,j}) = a\}$ (with repetition allowed), p_i is the natural projection from $\{n \mid \gcd(n, \alpha_i) = 1\}$ to U_{α_i} . Define $B_l(\alpha_i) = \{a \mid \theta_a(\alpha_i) = l\}$ for $l = 1, 2, \dots$, there are only finitely many l such that $B_l(\alpha_i) \neq \emptyset$. Let $C_l(\alpha_i) = \{b \in U_{\alpha_i} \mid ab \in B_l(\alpha_i), \forall a \in B_l(\alpha_i)\}$ if $B_l(\alpha_i) \neq \emptyset$ and $C_l(\alpha_i) = U_{\alpha_i}$ otherwise. Finally define $C(\alpha_i) = \bigcap_{l=1}^{\infty} C_l(\alpha_i)$, and $\bar{C}(\alpha_i) = p_i^{-1}(C(\alpha_i))$.

Theorem 3.6. $D(M(g; \frac{\beta_{1,1}}{\alpha_1}, \dots, \frac{\beta_{1,m_1}}{\alpha_1}, \dots, \frac{\beta_{n,1}}{\alpha_n}, \dots, \frac{\beta_{n,m_n}}{\alpha_n})) = \bigcap_{i=1}^n \bar{C}(\alpha_i).$

3.3. A brief comment of the topic and organization of the paper. Theorem 3.1 was appeared in [Wa1]. The proof of the "only if" part in Theorem 3.1 is based on the results on simplicial volume developed by Gromov, Thurston and Soma (see [So]), and various classical results by others on 3-manifold topology and group theory ([He], [SW], [R]). The proof of "if" part in Theorem 3.1 is a sequence elementary constructions, which were essentially known before, for example see [HL] and [KM] for (3). That graph manifolds admit no self-maps of degrees > 1 also follows from a recent work [De2].

The table in Theorem 3.2 is quoted from [Du], which generalizes the earlier work [HKWZ], which is presented as below.

Proposition 3.7. For 3-manifold M supporting S^3 geometry,

$$D_{iso}(M) = \{k^2 + l|\pi_1(M)|, \text{ where } k \text{ and } |\pi_1(M)| \text{ are co-prime}\}.$$

The topic of mapping degrees between (and to) 3-manifolds covered by S^3 has been discussed for long time and has much relation with other topics (see [Wa2] for details). We just mention several papers: in very old papers [Rh] and [Ol], the degrees of maps between any given pairs of lens spaces are obtained by using equivalent maps between spheres; in [HWZ], $D(M, L(p, q))$ can be computed for any 3-manifold M ; and in a recent one [MP], an algorithm (or formula) is given for the degrees of maps between given pairs of 3-manifolds covered by S^3 in term of their Seifert invariants.

Theorem 3.4 is proved in [SWW].

Theorem 3.3, Theorem 3.5 and Theorem 3.6 are proved in [SWWZ].

3.4. Some examples of computation.

Example 3.8. Let $M_1 = (P \# \bar{P}) \# (L(7, 1) \# L(7, 2) \# 2L(7, 3))$ and $M_2 = (2P \# \bar{P}) \# (L(7, 1) \# L(7, 2) \# L(7, 3))$, where P is the Poincare homology three sphere. Apply Theorem 3.3 we have

$$D(M_1) = \{840n + i \mid n \in \mathbb{Z}, i = 1, 71, 121, 169, 191, 239, 241, 289, 311, 359, 361,$$

409, 431, 479, 481, 529, 551, 599, 601, 649, 671, 719, 769, 839.}

$$D(M_2) = \{840n + i \mid n \in \mathbb{Z}, i = 1, 121, 169, 289, 361, 529.\}$$

Example 3.9. By Theorem 3.4, for the torus bundle M_ϕ , $\phi = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, among the first 20 integers > 0 , exactly $1, 4, 5, 9, 11, 16, 19, 20 \in D(M_\phi)$.

Example 3.10. For Nil 3-manifold $M = M(0; \frac{\beta_1}{2}, \frac{\beta_2}{3}, \frac{\beta_3}{6})$,

$$D(M) = \{l^2 \mid l = m^2 + mn + n^2, l \equiv 1 \pmod{6}, m, n \in \mathbb{Z}\}.$$

The numbers in $D(M)$ smaller than 10000 are exactly 1, 49, 169, 361, 625, 961, 1369, 1849, 2401, 3721, 4489, 5329, 6241, 8291, 9409.

Example 3.11. For $H^2 \times E^1$ manifold $M = M(2; \frac{1}{5}, \frac{1}{5}, -\frac{2}{5}, \frac{1}{7}, \frac{2}{7}, -\frac{3}{7})$, apply Theorem 3.6 we have $D(M) = \{5n + 1 \mid n \in \mathbb{Z}\} \cap \{7n + i \mid n \in \mathbb{Z}, i = 1, 2, 4\} = \{35n + i \mid n \in \mathbb{Z}, i = 1, 11, 16\}$.

4. REALIZATION OF SELF-MAP OF DEGREE ± 1 BY A HOMEOMORPHISMS

This section is based on [Sun].

Given a closed orientable n -manifold M , it is natural to ask, whether all the degree ± 1 self-maps on M can be homotopic to homeomorphisms. Without specific description, all the manifolds below are closed and orientable.

If the property stated above holds for M , we say M has property H. In particular, if all the degree 1 (-1) self-maps on M can be homotopic to homeomorphisms, we say M has property 1H ($-1H$). M has property H if and only if M has both property 1H and property $-1H$. We can observe that, if M admits an orientation-reversing self-homeomorphism, then M has property 1H if and only if M has property $-1H$. So we mostly only concern property 1H.

Below we would like to determine which prime 3-manifolds, which are the basic part of 3-manifolds, has property H.

It is known that each degree ± 1 self-map map f on M induces an isomorphism $f_* : \pi_1(M) \rightarrow \pi_1(M)$.

Hyperbolic 3-manifolds and Haken manifolds have property H by the celebrated Mostow rigidity theorem [M] and Waldhausen's theorem on Haken manifolds (see 13.6 of [He]).

This two theorems cover most cases of irreducible 3-manifolds, including: the manifolds with nontrivial JSJ decomposition, hyperbolic manifolds, Seifert manifolds M with incompressible surface. So the remaining cases are:

Class 1. manifolds supporting S^3 -geometry;

Class 2. Seifert manifolds supporting Nil or $\widetilde{PSL(2, R)}$ geometries with orbifold $S^2(p, q, r)$;

4.1. Main Results. Class 1. According to [Or] or [Sc], the fundamental group of a 3-manifold supporting S^3 -geometry is among the following eight types: \mathbb{Z}_p , D_{4n}^* , T_{24}^* , O_{48}^* , I_{120}^* , $T'_{8 \cdot 3^q}$, $D'_{n' \cdot 2^q}$ and $\mathbb{Z}_m \times \pi_1(N)$, where N is a S^3 3-manifold, $\pi_1(N)$ belongs to the

previous seven ones, and $|\pi_1(N)|$ is coprime to m . The cyclic group Z_p is realized by lens space $L(p, q)$, each group in the remaining types is realized by a unique S^3 -manifold.

Theorem 4.1. *For M supporting S^3 -geometry, M has property 1H if and only if M belongs to one of the following classes:*

- i) S^3 ;
- ii) $L(p, q)$ satisfies one of the following:
 - a) $p = 2, 4, p_1^{e_1}, 2p_1^{e_1}$;
 - b) $p = 2^s (s > 2), 4p_1^{e_1}, p_1^{e_1} p_2^{e_2}, 2p_1^{e_1} p_2^{e_2}, q^2 \equiv 1 \pmod{p}$ and $q \neq \pm 1$;
- iii) $\pi_1(M) = \mathbb{Z}_m \times D_{4k}^*$, $(m, k) = (1, 2^k), (p_1^{e_1}, 2), (1, p_2^{e_2})$ or $(p_1^{e_1}, p_2^{e_2})$;
- iv) $\pi_1(M) = D'_{2^{k+2}p_1^{e_1}}$;
- v) $\pi_1(M) = T_{24}^*$ or $\mathbb{Z}_{p_1^{e_1}} \times T_{24}^*$;
- vi) $\pi_1(M) = T'_{8 \cdot 3^{k+1}}$;
- vii) $\pi_1(M) = O_{48}^*$ or $\mathbb{Z}_{p_1^{e_1}} \times O_{48}^*$;
- viii) $\pi_1(M) = I_{120}^*$ or $\mathbb{Z}_{p_1^{e_1}} \times I_{120}^*$.

Where all the p_1, p_2 are odd prime numbers, e_1, e_2, k, m are positive integers.

By [HKWZ] and elementary number theory, among all the S^3 -manifolds, only S^3 and lens spaces admit degree -1 self-maps. When considering about property $-1H$, it is reasonable to restrict the manifold to be $L(p, q)$.

Proposition 4.2. *$L(p, q)$ has property $-1H$ if and only if $L(p, q)$ belongs to one of the following classes:*

- i) $4|p$ or some odd prime factor of p is in $4k + 3$ type;
- ii) $q^2 \equiv -1 \pmod{p}$ and $p = 2, p_1^{e_1}, 2p_1^{e_1}$, where p_1 is $4k + 1$ type prime number.

Essentially, it is known that the manifolds in Class 2 have property H. However, the author can't find a proper reference and he can just copy the proof of Theorem 3.9 of [Sc] to prove this result.

Theorem 4.3. *For Seifert manifolds M supporting Nil or $\widetilde{PSL(2, R)}$ geometries with orbifold $S^2(p, q, r)$, M has property H.*

Synthesize from Mostow and Waldhausen's theorem and Theorem 4.1, 4.3, Proposition 4.2, we get the following consequence:

Theorem 4.4. *Suppose M is a prime geometrizable 3-manifold.*

- 1) M has property 1H if and only if M belongs to one of the following classes:
 - i) M does not support S^3 -geometry;
 - ii) M is in one of the classes stated in Theorem 4.1
- 2) M has property $-1H$ if and only if M belongs to one of the following classes:
 - i) M does not support S^3 -geometry;
 - ii) M is in one of the classes stated in Proposition 4.2.
- 3) M has property H if and only if M belongs to one of the following classes:
 - i) M does not support S^3 -geometry;
 - ii) M is in one of the classes except ii) stated in Theorem 4.1;
 - iii) $L(p, q)$ satisfies one of the following:
 - a) $p = 2, 4$;
 - b) $p = p_1^{e_1}, 2p_1^{e_1}$, where p_1 is $4k + 3$ type prime number;

- c) $p = p_1^{e_1}, 2p_1^{e_1}$, where p_1 is $4k+1$ type prime number and $q^2 \equiv -1 \pmod{p}$;
 d) $p = 2^s (s > 2), 4p_1^{e_1}$, $q^2 \equiv 1 \pmod{p}$, $q \neq \pm 1$;
 e) $p = p_1^{e_1} p_2^{e_2}, 2p_1^{e_1} p_2^{e_2}$, where one of p_1, p_2 is $4k+3$ type prime number, $q^2 \equiv 1 \pmod{p}$, $q \neq \pm 1$.

Indeed the proof of above theorems in [Sun] give much stronger results. For simplicity, we only explain the situation for 1H.

Let $K(M) = \{\phi \in \text{Out}(\pi_1(M)) \mid \exists f : M \rightarrow M, f_* \in \phi, \deg(f) = 1\}$. It is known $K(M)$ is 1 – 1 corresponds with $\{\text{degree 1 self-maps } f \text{ on } M\}/\text{homotopy}$.

Let $K'(M) = \{\phi \in \text{Out}(\pi_1(M)) \mid \phi \text{ is realized by orientation preserving homeomorphism}\}$, which is a subgroup of $K(M)$. $K'(M)$ is 1 – 1 corresponds with $\mathcal{MCG}^+(M)$, the orientation preserving subgroup of mapping class group of M .

To determine whether M has property 1H, we need only determine whether $K(M) = K'(M)$, or whether $|K(M)| = |\mathcal{MCG}^+(M)|$. Define the realization coefficient of M to be

$$RC(M) = \frac{|K(M)|}{|K'(M)|}.$$

So M has property 1H if and only if $RC(M) = 1$. The $RC(M)$ is completely determined for each 3-manifold support S^3 -geometry in [Sun].

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